24. Area of Triangles

1 Congruent Triangles

Definition: Same shape and size, can be perfectly superimposed. Written as $\triangle ABC \cong \triangle DEF$

Conditions for Congruence

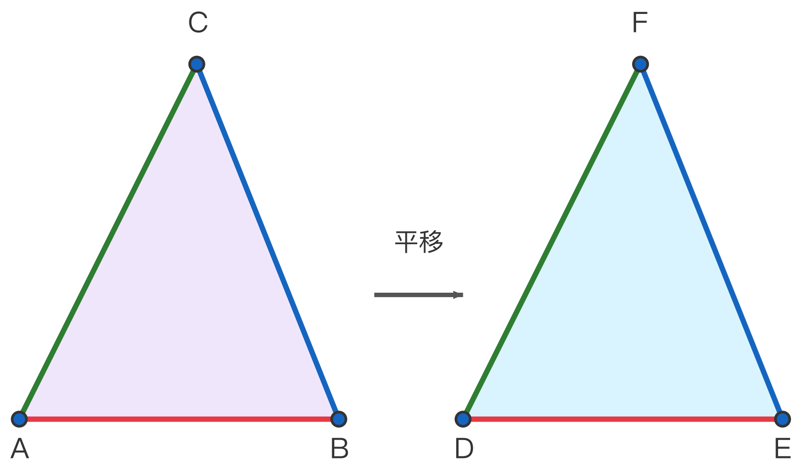
SSS — Three corresponding sides equal

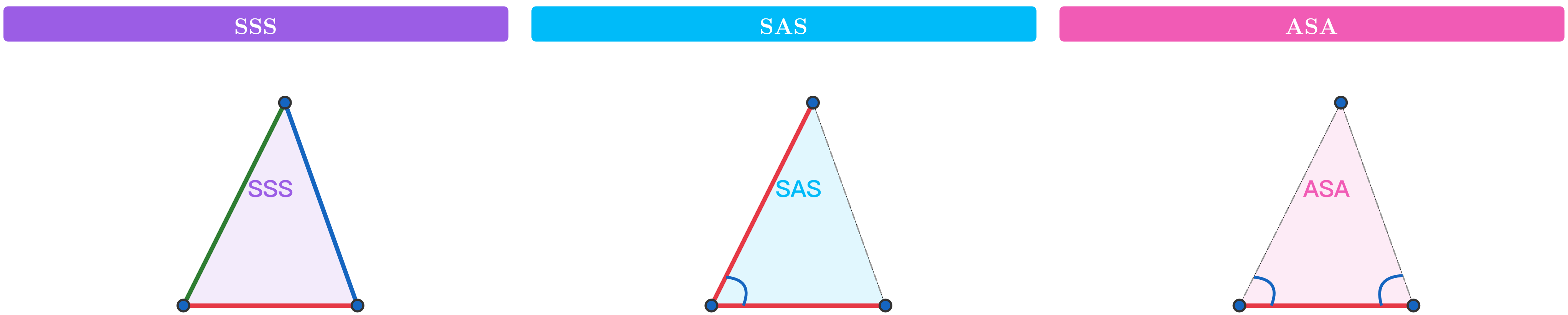
SAS — Two sides and their included angle equal

ASA — Two angles and their included side equal

AAS — Two angles and a non-included side equal (two angles equal implies the third angle is equal, equivalent to ASA)

Properties: Corresponding sides are equal, corresponding angles are equal





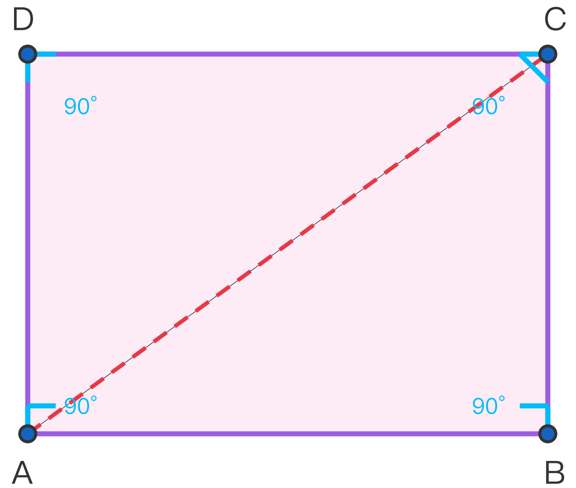
2 Rectangle

Definition: A quadrilateral with four right angles, $\angle A = \angle B = \angle C = \angle D = 90°$

Theorem: A diagonal divides a rectangle into two congruent triangles

> In $\triangle ABC$ and $\triangle CDA$:
> ① $AC = AC$ (common side)
> ② $AB = CD$, $AD = BC$ (rectangle opposite sides equal)
> ③ $\therefore \triangle ABC \cong \triangle CDA$ (SSS)
> $\therefore$ A rectangle is divided by its diagonal into two congruent right triangles

Area: Area of rectangle = **length × width** (fundamental fact)

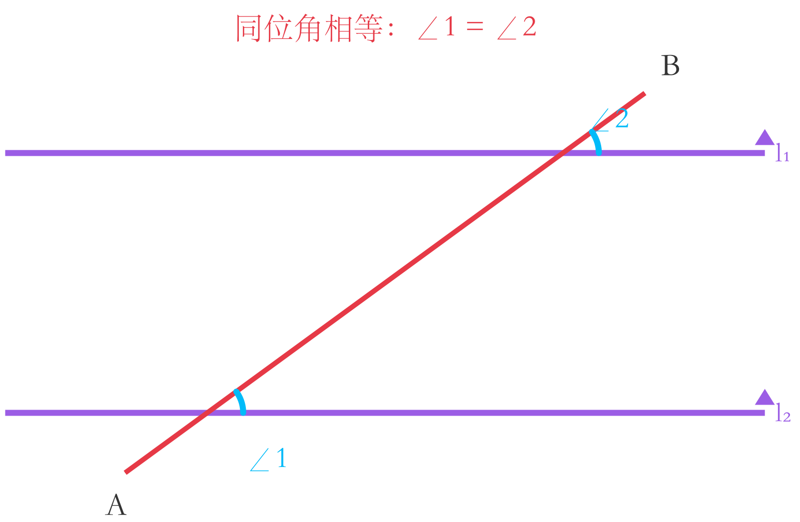


3 Corresponding Angles

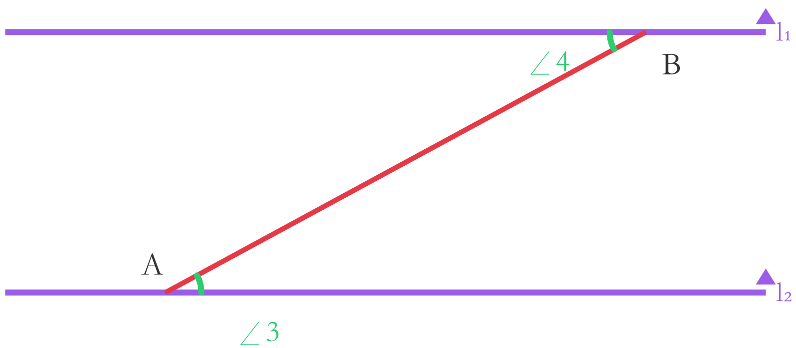
Theorem: When two parallel lines are cut by a transversal, corresponding angles are equal

> As shown, $l_1 \parallel l_2$, AB is the transversal
> $\angle 1$ and $\angle 2$ are in corresponding positions, called corresponding angles
> By the parallel lines property: $\angle 1 = \angle 2$

We will use this theorem to prove triangle congruence later



内错角相等：∠3 = ∠4

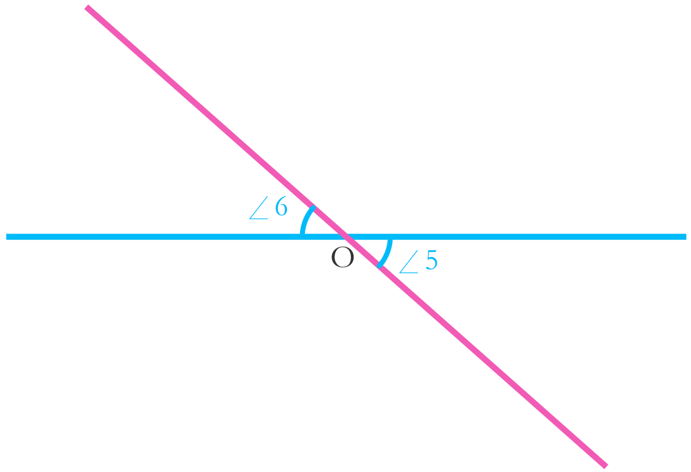


4 Alternate Interior Angles

Theorem: When two parallel lines are cut by a transversal, alternate interior angles are equal

> As shown, $l_1 \parallel l_2$, AB is the transversal
> ∠3 and ∠4 are on the inner side of the parallel lines, called alternate interior angles
> By the parallel lines property: ∠3 = ∠4

对顶角相等：∠5 = ∠6



5 Vertical Angles

Theorem: When two lines intersect, vertical angles are equal

> As shown, two lines intersect at O
> ∠5 and ∠6 are vertical angles
> By the vertical angles property: ∠5 = ∠6

6 Parallelogram Opposite Sides

Definition: A quadrilateral with both pairs of opposite sides parallel

Theorem: Opposite sides of a parallelogram are equal (AB = CD, AD = BC)

Proof: Connect diagonal AC

In $\triangle ABC$ and $\triangle CDA$:

① $\angle BAC = \angle DCA$ (AB // CD, alternate interior angles equal, §4)

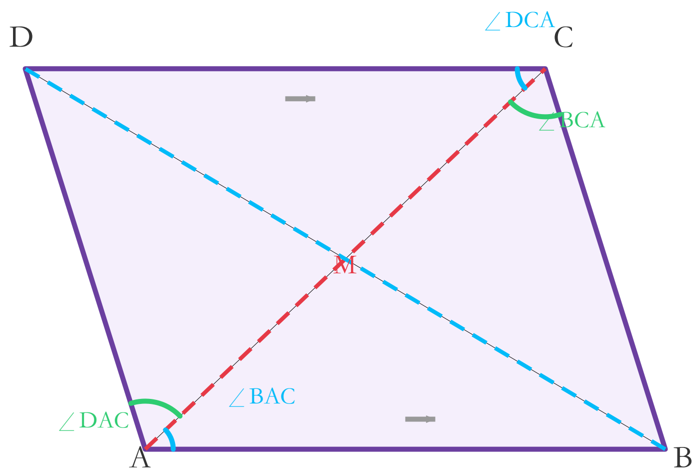
② $\angle BCA = \angle DAC$ (AD // BC, alternate interior angles equal, §4)

③ AC = AC (common side)

$\therefore$ $\triangle ABC \cong \triangle CDA$ (ASA)

$\therefore$ AB = CD, BC = DA

This theorem will be used in the next section to prove area



7 Area of a Parallelogram

Theorem: Area of parallelogram = base × height

Proof: From D, draw DE ⊥ AB at E; from C, draw CF ⊥ AB at F

In Rt△DEA and Rt△CFB:

① AD = BC (§6 parallelogram opposite sides equal)

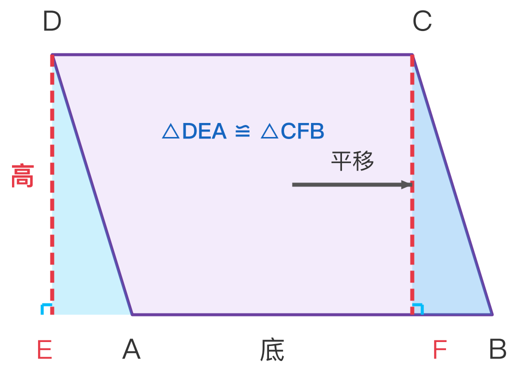
② ∠DAE = ∠CBF (AD // BC, §3 corresponding angles equal)

③ ∠DEA = ∠CFB = 90° (construction)

∴ △DEA ≌ △CFB (AAS)

∴ $S_{\triangle DEA} = S_{\triangle CFB}$

$$
\begin{aligned}
S_{▱ABCD} &= S_{\text{trapezoid DEAB}} + S_{\triangle DEA} \\
&= S_{\text{trapezoid DEAB}} + S_{\triangle CFB} \\
&= S_{\text{rectangle DEFC}} \\
&= DE \times EF = \text{height} \times \text{base}
\end{aligned}
$$



Area of Parallelogram = base × height

base: Any side of the parallelogram (e.g., AB) height: Perpendicular distance from opposite side to the base (e.g., DE)

24. Area of Triangles

8 Triangle Area Formula

Theorem: Area of triangle = ½ × base × height

Proof: Given $\triangle ABC$, draw a line through A parallel to BC,
and a line through C parallel to AB; they intersect at D
$\therefore$ ABCD is a parallelogram (by definition)

In $\triangle ABC$ and $\triangle CDA$:
① $AC = AC$ (common side)
② $AB = CD$, $BC = DA$ (§6 parallelogram opposite sides equal)
$\therefore \triangle ABC \cong \triangle CDA$ (SSS)

$\therefore S_{\triangle ABC} = S_{\triangle CDA} = \frac{1}{2} \times S_{\square ABCD}$
By §7, $S_{\square ABCD} = \text{base} \times \text{height}$

$\therefore S_{\triangle ABC} = \frac{1}{2} \times \text{base} \times \text{height}$

$$S_{\triangle} = \frac{1}{2} \times \text{base} \times \text{height}$$

base: Any side of the triangle (e.g., AB) height: Perpendicular distance from opposite vertex to the base (e.g., CH, where H is the foot of perpendicular)